J. Fluid Mech. (1985), *vol.* 159, *pp.* 241–258
Printed in Great Britain

Non-parallel-flow stability of a two-dimensional buoyant plume

By SHUNICHI WAKITANI
Department of Mechanical Engineering, University of Osaka Prefecture, Sakai, Osaka 591, Japan

(Received 13 December 1983 and in revised form 23 April 1985)

The linear stability of a two-dimensional buoyant plume is analysed by taking into account the transverse velocity component and the streamwise variations of the basic flow and of the disturbance waves. The solutions indicate the dependence of the spatial amplification rate and wavenumber on the disturbance flow quantity under consideration as well as on the streamwise and transverse coordinates. The use of the flow quantity relative to the basic flow leads to close agreement with the neutral curve according to quasi-parallel-stability theory, the usual method for treating nearly parallel flows. However, the amplification rate within an unstable region shows substantial deviation from that predicted by the quasi-parallel theory. The validity of this non-parallel theory is supported by the existing experimental data.

1. Introduction

The stability of laminar flows has attracted the attention of many researchers. The quasi-parallel-flow assumption has been used extensively in linear-stability analyses for nearly parallel shear flows of boundary-layer type. However, it is only comparatively recently that the linear stability of natural-convection boundary layers has received attention, since the effect of velocity–temperature disturbance coupling through the buoyancy term in momentum equation increases the complexity.

The stability characteristics of a buoyant plume generated above a heated body are substantially different from those of flows arising adjacent to the surface, owing to the absence of damping of the disturbance at the surface. In general, unbounded flows such as plumes and jets are more unstable than bounded or semi-bounded flows. The quasi-parallel theory results in very low values of critical Reynolds numbers for many unbounded flows. It is not clearly justifiable for these flows to treat the basic flow as quasi-parallel, since the transverse velocity component and streamwise variations of the basic flow are not negligible at such low Reynolds numbers.

This paper concerns the linear stability of a steady, buoyant plume which is idealized as a two-dimensional plume above a horizontal line source of heat in an extensive medium. The quasi-parallel theory has been applied to such a plume by Pera & Gebhart (1971) and Wakitani & Yosinobu (1984), who took into account the effect of the disturbance coupling. These results show that the plume is unstable at almost all Grashof numbers. The neutral curve obtained for a Prandtl number of 0.7 (air) does not exhibit the existence of a lower branch and a critical Grashof number. For the extremely low values of the Grashof number involved, the quasi-parallel-flow assumption is of questionable validity and a similar question could also be raised concerning the basic flow itself under the usual boundary-layer approximation.

The main difficulty in treating linear-stability problems for non-parallel flows lies in solving partial differential equations, which are non-separable, rather than the ordinary differential equations which occur for truly parallel flows.

Haaland & Sparrow (1973) accounted for only some non-parallel effects on the linear stability of the plume by retaining the transverse velocity component and the streamwise derivatives of the basic flow, excluded in the quasi-parallel approach. In this model the governing equations are separable and they reduce to the modified forms of the Orr–Sommerfeld equation with the temperature-coupling term and the disturbance-energy equation. The retention of the transverse velocity component leads to containment of the disturbance vorticity and temperature within the boundary layer (Haaland 1972). Haaland & Sparrow (1973) succeeded in obtaining a complete neutral curve, which exhibited both a lower branch and a critical Grashof number of about 5.1, in terms of the present notation. Their results show that the unstable region is smaller than that obtained from the quasi-parallel theory. However, Ling & Reynolds (1973) pointed out that the argument for deriving the modified Orr–Sommerfeld equation was contradictory.

Another contribution to the stability theory of a two-dimensional plume has been made by Hieber & Nash (1975). They analysed the linear stability by means of a systematic expansion which allows the incorporation of higher-order effects of the boundary layer into the basic flow. However, the lowest-order equation which they used is no more than the inviscid Orr–Sommerfeld equation; the viscous and temperature-coupling terms appear in the next-order equation together with those of some non-parallel effects. Their analysis leads to a large reduction of instability and yields a critical Grashof number of 7.3.

Thus, to the author's knowledge, no systematic solution has been developed to describe properly all non-parallel effects on the stability of a two-dimensional plume.

Recent developments in the linear-stability theory of nearly parallel flows have used the method of multiple scales, the WKB, ray, or slowly varying approximation method. This method was first applied to the Blasius boundary layer (Bouthier 1972, 1973; Gaster 1974; Saric & Nayfeh 1975). It results in the rather striking situation that the various disturbance quantities considered have different amplification rates, in comparison with the quasi-parallel theory. The stream function, for example, may be growing at some point in the flow while the velocity components may be decaying. This implies that the neutral curve will depend on the disturbance quantity considered. However, Saric & Nayfeh (1975) neglected to include the streamwise variation of the eigenfunction in the expression for the amplification rate (Eagles & Weissman 1975). Therefore, the much better agreement which they obtained with the existing experimental data than Gaster (1974) is largely accidental.

Several attempts to apply the method of multiple scales to the theory of the non-parallel stability of unbounded flows have been made for the Bickley jet (Garg & Round 1978; Garg 1981; Morris 1981). It is well known that the quasi-parallel theory yields a low value of 4.0 for the critical Reynolds number for this flow. Garg (1981) calculated the amplification rate based on the kinetic energy, averaged over time and integrated across the jet. His theory leads to an increase in the critical value from 4.0 to 21.6, but this seems to be opposite to the result of Garg & Round (1978) based on the same definition of the amplification rate as that used by Saric & Nayfeh (1975). Thus, in unbounded flows, the various definitions of the amplification rate may result in remarkably different stability characteristics. If one attempts to compare the theory with experimental data, one must therefore use the amplification rate based on the same quantity as that which was observed.

In this paper we follow Gaster (1974) and present a non-parallel-stability analysis which accounts for the vertical structure of the quasi-parallel solution for a two-dimensional plume in ordering the terms in the expansion. This analysis determines the solution as a function of the streamwise coordinate and estimates the effect of the streamwise variation of the plume basic flow on its stability characteristics. The results obtained for a Prandtl number of 0.7 are compared with the available experimental data.

2. Analysis

We consider the stability of a two-dimensional, steady plume above a horizontal line source of heat described by the stream function $\Psi(x, y)$ and temperature $T(x, y)$. Here x is the vertical coordinate measured from the heat source and y is normal to it. According to the standard procedure in linear-stability theory of superimposing a small disturbance upon the basic flow, the stream function and temperature of the disturbed flow are taken to have the form

$$\left.\begin{aligned}\tilde{\psi}(x, y, \bar{\tau}) &= \Psi(x, y) + \psi(x, y, \bar{\tau}),\\ \tilde{t}(x, y, \bar{\tau}) &= T(x, y) + t(x, y, \bar{\tau}),\end{aligned}\right\} \tag{1}$$

where $\bar{\tau}$ is time. Substituting (1) into the Navier–Stokes and energy equations based on the Boussinesq approximation, subtracting the equations for the basic flow and neglecting the nonlinear terms in the disturbance quantities, we obtain

$$\frac{\partial}{\partial \bar{\tau}}(\nabla^2\psi) + \frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) + \frac{\partial}{\partial x}(\nabla^2\Psi)\frac{\partial \psi}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) - \frac{\partial}{\partial y}(\nabla^2\Psi)\frac{\partial \psi}{\partial x} = g\beta^*\frac{\partial t}{\partial y} + \nu\nabla^4\psi, \tag{2}$$

and
$$\frac{\partial t}{\partial \bar{\tau}} + \frac{\partial \Psi}{\partial y}\frac{\partial t}{\partial x} + \frac{\partial T}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial t}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial \psi}{\partial x} = \kappa\nabla^2 t, \tag{3}$$

where g is the acceleration due to gravity, β^* the coefficient of thermal expansion, ν the kinematic viscosity and κ the thermal diffusivity.

Initial disturbance is assumed to exist at a location x_0 and solutions are sought for $x > x_0$. We introduce new coordinates (ξ, η) to rescale the problem (Gaster 1974):

$$\xi = \frac{x}{x_0}, \quad \eta = \frac{G^*}{x}y = \frac{y}{\lambda G^{\frac{2}{3}}\xi^{\frac{2}{5}}},$$

where
$$G^* \equiv \left(\frac{g\beta^* x^3 Q}{k\nu^2}\right)^{\frac{1}{5}} = G\xi^{\frac{3}{5}}, \quad \lambda = \left(\frac{k\nu^2}{g\beta^* Q}\right)^{\frac{1}{3}},$$

G^* being the modified Grashof number, k the thermal conductivity and Q the strength of heat source per unit length. In terms of ξ and η, the spatial derivatives transform into

$$\frac{\partial}{\partial x} = \frac{1}{\lambda G^{\frac{5}{3}}}\left(\frac{\partial}{\partial \xi} - \frac{2}{5}\frac{\eta}{\xi}\frac{\partial}{\partial \eta}\right), \quad \frac{\partial}{\partial y} = \frac{1}{\lambda G^{\frac{2}{3}}\xi^{\frac{2}{5}}}\frac{\partial}{\partial \eta}.$$

For the plume flow the stream function and temperature of the basic flow can be obtained as series of the form

$$\left.\begin{aligned}\Psi &= \nu\{G\xi^{\frac{3}{5}}f_0(\eta) + f_1(\eta) + O(G^{-\frac{2}{3}})\},\\ T - T_\infty &= \left(\frac{Q}{k}\right)(G\xi^{\frac{3}{5}})^{-1}\{h_0(\eta) + (G\xi^{\frac{3}{5}})^{-1}h_1(\eta) + O(G^{-\frac{5}{3}})\}\end{aligned}\right\} \tag{4}$$

by including higher-order terms of the boundary layer (Hieber & Nash 1975). Here T_∞ is the ambient temperature, assumed to be constant. The governing equations for f_0, f_1, h_0 and h_1 are given in appendix A.

For the disturbances we seek constant-frequency solutions of the form

$$\left.\begin{aligned} \psi &= \nu G\xi^{\frac{3}{5}}\phi(\xi,\eta)\,\mathrm{e}^{\mathrm{i}\theta}, \quad t = \left(\frac{Q}{k}\right)(G\xi^{\frac{3}{5}})^{-1}\,s(\xi,\eta)\,\mathrm{e}^{\mathrm{i}\theta} \\ \text{with} \qquad \theta &= G\int_1^{\xi} \xi^{-\frac{2}{5}}\alpha(\xi)\,\mathrm{d}\xi - \beta\tau. \end{aligned}\right\} \tag{5}$$

Here β is the real dimensionless frequency of the disturbance, α the complex dimensionless wavenumber as the separation parameter and τ the dimensionless time, based on the scales $\lambda G^{\frac{2}{3}}\xi^{\frac{2}{5}}$, $\lambda^2 G^{\frac{1}{3}}\xi^{\frac{1}{5}}/\nu$ for length and time respectively. Thus the real part of α defines the wavelength in terms of the local plume thickness.

Substituting (4) and (5) into (2) and (3) yields the equations for ϕ and s:

$$\begin{aligned} \mathrm{L}_1\phi + (G\xi^{\frac{3}{5}})^{-1}\Big\{\mathrm{D}s - (2\alpha\beta - 3\alpha^2 f_0' - f_0''')\left(\xi\frac{\partial\phi}{\partial\xi} - \tfrac{2}{5}\eta\,\mathrm{D}\phi + \tfrac{3}{5}\phi\right) - (\beta - 3\alpha f_0')\left(\xi\frac{\mathrm{d}\alpha}{\mathrm{d}\xi} - \tfrac{2}{5}\alpha\right)\phi \\ - \mathrm{i}\alpha[f_1'(\mathrm{D}^2-\alpha^2) - f_1''']\phi + \tfrac{1}{5}[(f_0'' + 2\eta f_0''')\,\mathrm{D}\phi + (3f_0 - 2\eta f_0')(\mathrm{D}^2-\alpha^2)\,\mathrm{D}\phi] \\ - f_0'\left[\xi\,\mathrm{D}^2\left(\frac{\partial\phi}{\partial\xi}\right) - \tfrac{2}{5}\eta\,\mathrm{D}^3\phi - \tfrac{1}{5}\mathrm{D}^2\phi\right]\Big\} = O(G^{-\frac{5}{3}}), \end{aligned} \tag{6}$$

$$\begin{aligned} \mathrm{L}_2 s + \mathrm{i}\alpha h_0'\phi - (G\xi^{\frac{3}{5}})^{-1}\Big\{\xi\left(f_0'\frac{\partial s}{\partial\xi} - h_0'\frac{\partial\phi}{\partial\xi}\right) + \mathrm{i}\alpha(f_1' s - h_1'\phi) \\ - \tfrac{3}{5}(h_0\,\mathrm{D}\phi + h_0'\phi + f_0\,\mathrm{D}s + f_0' s)\Big\} = O(G^{-\frac{5}{3}}), \end{aligned} \tag{7}$$

where

$$\mathrm{L}_1 \equiv (G\xi^{\frac{3}{5}})^{-1}(\mathrm{D}^2-\alpha^2)^2 - \mathrm{i}\alpha\left[\left(f_0' - \frac{\beta}{\alpha}\right)(\mathrm{D}^2-\alpha^2) - f_0'''\right],$$

$$\mathrm{L}_2 \equiv (Pr\,G\xi^{\frac{3}{5}})^{-1}(\mathrm{D}^2-\alpha^2) - \mathrm{i}\alpha\left(f_0' - \frac{\beta}{\alpha}\right).$$

The primes and $\mathrm{D} \equiv \partial/\partial\eta$ indicate differentiation with respect to η, and $Pr \equiv \nu/\kappa$ is the Prandtl number.

The plume flow is known to be more unstable to the disturbance for the case when ϕ is even and s is odd (Pera & Gebhart 1971). We shall consider only this mode of the disturbance and therefore take the boundary conditions as

$$\left.\begin{aligned} \mathrm{D}\phi = \mathrm{D}^3\phi = s = 0 \quad &\text{at } \eta = 0, \\ \phi,\ \mathrm{D}\phi,\ s \to 0 \quad &\text{as } \eta \to \infty. \end{aligned}\right\} \tag{8}$$

3. Solution

Equations (6) and (7) do not separate unless some terms of order G^{-1} are ignored. Solutions can be found of the form

$$\left.\begin{aligned} \phi(\xi,\eta) &= A(\xi)\,\phi_0(\eta;\xi) + \epsilon\phi_1(\xi,\eta) + o(\epsilon), \\ s(\xi,\eta) &= A(\xi)\,s_0(\eta;\xi) + \epsilon s_1(\xi,\eta) + o(\epsilon), \end{aligned}\right\} \tag{9}$$

where A is a weak function of ξ incorporated to take some account of the streamwise variations of the wavenumber and eigenfunctions. A is unknown at this level of approximation. Since the approximation used in deriving the leading terms in (9) neglects terms of order G^{-1} it may be expected that ϵ is of order G^{-1}. But the viscous and temperature-coupling terms in (6), and the diffusion term in (7) should be retained in the lowest-order approximation so that the resulting equations reduce to those of the Orr–Sommerfeld type derived from the quasi-parallel theory.

Substituting (9) into (6) and (7), and putting $\epsilon = (G\xi^{\frac{3}{5}})^{-1}$ we obtain the following.

$$O(\epsilon^0): \qquad \mathbf{L}\boldsymbol{\Phi}_0 = 0, \tag{10}$$

$$O(\epsilon^1): \qquad \mathbf{L}\boldsymbol{\Phi}_1 = \boldsymbol{M}, \tag{11}$$

where
$$\mathbf{L} \equiv \begin{pmatrix} \mathrm{L}_1 & G^{-1}\xi^{-\frac{3}{5}}\mathrm{D} \\ \mathrm{i}\alpha h_0' & \mathrm{L}_2 \end{pmatrix}, \quad \boldsymbol{\Phi}_j \equiv \begin{pmatrix} \phi_j \\ s_j \end{pmatrix} \quad \text{with } j = 0, 1,$$

and
$$\boldsymbol{M} \equiv \begin{pmatrix} F_1 A + F_2 \xi \dfrac{\mathrm{d}A}{\mathrm{d}\xi} \\ F_3 A + F_4 \xi \dfrac{\mathrm{d}A}{\mathrm{d}\xi} \end{pmatrix},$$

the Fs being known and defined in appendix B. From (8) the boundary conditions are

$$\left.\begin{array}{ll} \mathrm{D}\phi_j = \mathrm{D}^3\phi_j = s_j = 0 & \text{at } \eta = 0, \\ \phi_j, \mathrm{D}\phi_j, s_j \to 0 & \text{as } \eta \to \infty \end{array}\right\} \tag{12}$$

for $j = 0$ and 1. Noting that $G^* = G\xi^{\frac{3}{5}}$ we find that the eigenvalue problem in (10) and (12) is the familiar Orr–Sommerfeld problem based on the quasi-parallel assumption (Pera & Gebhart 1971). In addition, (11) and (12) describe the non-parallel effects.

For the solution of the inhomogeneous problem consisting of (11) and (12), the solvability condition can be applied by use of the adjoint function $\boldsymbol{\Phi}^*$. This solvability condition is given by

$$\int_0^\infty {}^{\mathrm{T}}\boldsymbol{\Phi}^* \boldsymbol{M}\,\mathrm{d}\eta = 0, \tag{13}$$

where ${}^{\mathrm{T}}\boldsymbol{\Phi}^* \equiv (\phi^*, s^*)$ denotes the transpose of $\boldsymbol{\Phi}^*$. $\boldsymbol{\Phi}^*$ is the eigenfunction corresponding to the eigenvalue α of the adjoint problem and therefore satisfies the adjoint equation

$$\mathbf{L}^*\boldsymbol{\Phi}^* = 0, \tag{14}$$

and boundary conditions identical with (12)

$$\left.\begin{array}{ll} \mathrm{D}\phi^* = \mathrm{D}^3\phi^* = s^* = 0 & \text{at } \eta = 0, \\ \phi^*, \mathrm{D}\phi^*, s^* \to 0 & \text{as } \eta \to \infty, \end{array}\right\} \tag{15}$$

where
$$\mathbf{L}^* \equiv \begin{pmatrix} \mathrm{L}_1^* & \mathrm{i}\alpha h_0' \\ -G^{-1}\xi^{-\frac{3}{5}}\mathrm{D} & \mathrm{L}_2 \end{pmatrix}$$

with
$$\mathrm{L}_1^* \equiv (G\xi^{\frac{3}{5}})^{-1}(\mathrm{D}^2-\alpha^2)^2 - \mathrm{i}\alpha\left[\left(f_0' - \frac{\beta}{\alpha}\right)(\mathrm{D}^2-\alpha^2) + 2f_0''\,\mathrm{D}\right].$$

The adjoint problem has the same eigenvalue as the original problem. For the inhomogeneous equation (11) a solution satisfying the condition (12) is impossible unless the solvability condition (13) is satisfied. We must therefore take

$$\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}=\frac{-\int_0^\infty (F_1\phi^*+F_3 s^*)\,\mathrm{d}\eta}{\int_0^\infty (F_2\phi^*+F_4 s^*)\,\mathrm{d}\eta}. \tag{16}$$

The first correction to the parallel-flow solution $A(\xi)$, which would be a constant for a truly parallel basic flow, is obtained from (16) for all $\xi > 1$.

In order to determine $\mathrm{d}A/\mathrm{d}\xi$ from (16), we need to evaluate $\partial\boldsymbol{\Phi}_0/\partial\xi$ and $\mathrm{d}\alpha/\mathrm{d}\xi$. To do this we differentiate (10) with respect to ξ, and can readily write the result as

$$\mathbf{L}\left(\xi\frac{\partial\boldsymbol{\Phi}_0}{\partial\xi}\right)=\begin{pmatrix} g_1+\xi\dfrac{\mathrm{d}\alpha}{\mathrm{d}\xi}g_2 \\ g_3+\xi\dfrac{\mathrm{d}\alpha}{\mathrm{d}\xi}g_4\end{pmatrix}, \tag{17}$$

where the g's are known functions of ϕ_0 and s_0. The boundary conditions are also obtained by differentiating (12) with respect to ξ.

This inhomogeneous problem is also solved by applying the solvability condition that yields

$$\xi\frac{\mathrm{d}\alpha}{\mathrm{d}\xi}=\frac{-\int_0^\infty (g_1\phi^*+g_3 s^*)\,\mathrm{d}\eta}{\int_0^\infty (g_2\phi^*+g_4 s^*)\,\mathrm{d}\eta}. \tag{18}$$

With $\xi\,\mathrm{d}\alpha/\mathrm{d}\xi$ known from (18), $\xi\,\partial\boldsymbol{\Phi}_0/\partial\xi$ can be evaluated from integration of (17).

4. Numerical procedure

The sixth-order system of linear homogeneous differential equations (10) with the boundary conditions (12) is first solved using the method of Hieber & Gebhart (1971). This solution is written as the sum of three linearly independent solutions:

$$\boldsymbol{\Phi}_0=\boldsymbol{\Phi}_{01}+B_2(\xi)\,\boldsymbol{\Phi}_{02}+B_3(\xi)\,\boldsymbol{\Phi}_{03}, \tag{19}$$

where $\boldsymbol{\Phi}_{0j}\equiv\begin{pmatrix}\phi_{0j}\\ s_{0j}\end{pmatrix}$ with $j = 1, 2, 3$

and the coefficient of $\boldsymbol{\Phi}_{01}$ taken to be 1, thereby fixing the arbitrary scale of the disturbance. The asymptotic solutions as $\eta\to\infty$ of the three linearly independent solutions $\boldsymbol{\Phi}_{01}$, $\boldsymbol{\Phi}_{02}$ and $\boldsymbol{\Phi}_{03}$ are given by Pera & Gebhart (1971) and Wakitani & Yosinobu (1984). These asymptotic solutions are used as starting values for the numerical integration of (10), which proceeds inward from some large value of $\eta(=\eta_e)$ to the centreline of the plume ($\eta = 0$). For given values of Pr, β and G^* ($=G\xi^{\frac{2}{5}}$), a complex value for α is guessed and $\boldsymbol{\Phi}_{01}$, $\boldsymbol{\Phi}_{02}$ and $\boldsymbol{\Phi}_{03}$ are integrated separately using a fourth-order Runge–Kutta method. From the values of the three solutions at $\eta = 0$, B_2 and B_3 are determined from the boundary conditions $\mathrm{D}\phi_0(0) = \mathrm{D}^3\phi_0(0) = 0$. The remaining condition, i.e. $s_0(0) = 0$, is satisfied for the given β and G^* if, and only if, α takes an eigenvalue. The initial guess for α is refined iteratively by applying the Newton–Raphson method to the unsatisfied boundary condition. The process is repeated until $|s_0(0)|$ is sufficiently small ($\leqslant 10^{-5}$, typically).

With the eigenvalue α obtained, (14) is integrated to determine the adjoint function Φ^* using a procedure similar to the one above. No iteration is needed since the adjoint problem has the same eigenvalues. Hence the calculation of Φ^* was used as a check on the accuracy of the calculated eigenvalues.

Then $\xi\, \mathrm{d}\alpha/\mathrm{d}\xi$ is evaluated from (18), and the inhomogeneous equation (17) is integrated to determine $\xi\, \partial\Phi_0/\partial\xi$ by using the proper starting values.

With α, $\xi\, \mathrm{d}\alpha/\mathrm{d}\xi$, Φ_0, Φ^* and $\xi\, \partial\Phi_0/\partial\xi$ known, $(\xi/A)\, \mathrm{d}A/\mathrm{d}\xi$ is calculated from (16). All computations were performed in double precision on an ACOS 700 computer. The effect of different mesh size $\Delta\eta$ and values of η_e was examined. Typical values used were $\Delta\eta = 0.1$ and $\eta_e = 10$ (for small βG^*, $\eta_e = 24$).

5. The amplification rates and wavenumbers

In truly parallel flows the eigenfunctions are independent of the streamwise location ξ, and the exponential part of the stream function or temperature uniquely defines the wavenumber and amplification rate. These parameters are, however, not uniquely defined for non-parallel flows. The eigenfunctions vary slowly with the streamwise location, and the wavenumber and amplification rate therefore depend on the disturbance quantities considered. The choice of different values for these quantities leads to results differing by amounts of order G^{-1}.

The 'physical' amplitudes of $u = \partial\psi/\partial y$ and t, which may be measured simply in an experiment, are given by

$$\left.\begin{aligned} |u| &= \left(\frac{\nu}{\lambda}\right) G^{\frac{1}{3}} \xi^{\frac{1}{5}} [|A|\,|\mathrm{D}\phi_0|\, \mathrm{e}^{-\theta_i} + O(G^{-1})], \\ |t| &= \left(\frac{Q}{k}\right) (G\xi^{\frac{3}{5}})^{-1} [|A|\,|s_0|\, \mathrm{e}^{-\theta_i} + O(G^{-1})], \end{aligned}\right\} \tag{20}$$

where θ_i is the imaginary part of θ. We define the amplification rates for u and t as

$$\left.\begin{aligned} K(u) &\equiv \frac{\xi^{\frac{3}{5}}}{G} \frac{\partial}{\partial\xi} \ln|u| = -\alpha_i + (G\xi^{\frac{3}{5}})^{-1} \left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \left(\frac{\xi}{\mathrm{D}\phi_0}\frac{\partial \mathrm{D}\phi_0}{\partial\xi}\right)_r + \frac{1}{5}\right], \\ K(t) &\equiv \frac{\xi^{\frac{2}{5}}}{G} \frac{\partial}{\partial\xi} \ln|t| = -\alpha_i + (G\xi^{\frac{3}{5}})^{-1} \left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \left(\frac{\xi}{s_0}\frac{\partial s_0}{\partial\xi}\right)_r - \frac{3}{5}\right] \end{aligned}\right\} \tag{21}$$

to $O(G^{-1})$. Here suffixes r and i denote real and imaginary parts. The term $\xi^{\frac{3}{5}}/G$ is included in the definition so that the leading terms in (21) agree with the amplification rate given by the quasi-parallel theory: $-\alpha_i$. The terms in the group of order G^{-1} arise from the amplitude function, the eigenvalue modification with Grashof number and the coordinate system respectively. When α_i is equal to zero, i.e. at the neutral point determined by the quasi-parallel theory, there is still growth or decay due to the higher-order effects. Since the second terms in the group $O(G^{-1})$ depend on η, their values must be evaluated at some position, e.g. where $|u|$ or $|t|$ is a maximum, to determine the amplification rate.

It is convenient to introduce a measure of the amplitude relative to the basic flow, since the basic state is changing downstream (Eagles & Weissman 1975). We define the 'relative' amplitudes of u and t as $|\hat{u}| = |u|/U_0$ and $|\hat{t}| = |t|/(T_0 - T_\infty)$ respectively, where U_0 and T_0 are the streamwise velocity component and temperature of the basic flow on the plume centreline ($\eta = 0$) respectively, and therefore

$$
\left.\begin{aligned}
K(\hat{u}) &= -\alpha_i + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \left(\frac{\xi}{\mathrm{D}\phi_0}\frac{\partial \mathrm{D}\phi_0}{\partial \xi}\right)_r\right],\\
K(\hat{t}) &= -\alpha_i + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \left(\frac{\xi}{s_0}\frac{\partial s_0}{\partial \xi}\right)_r\right]
\end{aligned}\right\} \tag{22}
$$

to $O(G^{-1})$.

Furthermore, we use the kinetic-energy and thermal-energy integrals to characterize the stability and thus remove any ambiguity in the definition of the neutral curve. These quantities are difficult to measure but physically meaningful parameters. They are defined as

$$
E \equiv \int_0^\infty (\overline{u^2} + \overline{v^2})\,\mathrm{d}y, \quad H \equiv \int_0^\infty \overline{t^2}\,\mathrm{d}y,
$$

where an overbar indicates an average over a period. The amplification rates based on these quantities are

$$
\left.\begin{aligned}
K(E) &\equiv \frac{\xi^{\frac{2}{5}}}{2G}\frac{\mathrm{d}}{\mathrm{d}\xi}\ln E = -\alpha_i + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \frac{\xi}{2e}\frac{\mathrm{d}e}{\mathrm{d}\xi} + \frac{2}{5}\right],\\
K(H) &\equiv \frac{\xi^{\frac{2}{5}}}{2G}\frac{\mathrm{d}}{\mathrm{d}\xi}\ln H = -\alpha_i + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \frac{\xi}{2h}\frac{\mathrm{d}h}{\mathrm{d}\xi} - \frac{2}{5}\right]
\end{aligned}\right\} \tag{23}
$$

to $O(G^{-1})$, where

$$
e \equiv \int_0^\infty (|\mathrm{D}\phi_0|^2 + |\alpha|^2|\phi_0|^2)\,\mathrm{d}\eta, \quad h \equiv \int_0^\infty |s_0|^2\,\mathrm{d}\eta.
$$

We also define the relative kinetic-energy and thermal-energy integrals as

$$
\hat{E} \equiv E\Big/\int_0^\infty \left(\frac{\partial \Psi}{\partial y}\right)^2 \mathrm{d}y, \quad \hat{H} \equiv H\Big/\int_0^\infty (T - T_\infty)^2\,\mathrm{d}y,
$$

and therefore

$$
\left.\begin{aligned}
K(\hat{E}) &= -\alpha_i + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \frac{\xi}{2e}\frac{\mathrm{d}e}{\mathrm{d}\xi}\right],\\
K(\hat{H}) &= -\alpha_i + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_r + \frac{\xi}{2h}\frac{\mathrm{d}h}{\mathrm{d}\xi}\right]
\end{aligned}\right\} \tag{24}
$$

to $O(G^{-1})$.

In addition to the amplification rate we consider higher-order corrections to the wavenumbers. The phases of u and t are given by

$$
\left.\begin{aligned}
\mathrm{ph}\,u &= \theta_r + \arg A + \arg\,\mathrm{D}\phi_0 + O(G^{-1}),\\
\mathrm{ph}\,t &= \theta_r + \arg A + \arg s_0 + O(G^{-1}),
\end{aligned}\right\} \tag{25}
$$

where arg denotes the argument. The wavenumber is then the derivative of the phase of the disturbance with respect to ξ. We therefore define the wavenumbers for u and t as

$$
\left.\begin{aligned}
N(u) &\equiv \frac{\xi^{\frac{2}{5}}}{G}\frac{\partial}{\partial \xi}\mathrm{ph}\,u = \alpha_r + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_i + \left(\frac{\xi}{\mathrm{D}\phi_0}\frac{\partial \mathrm{D}\phi_0}{\partial \xi}\right)_i\right],\\
N(t) &\equiv \frac{\xi^{\frac{2}{5}}}{G}\frac{\partial}{\partial \xi}\mathrm{ph}\,t = \alpha_r + (G\xi^{\frac{2}{5}})^{-1}\left[\left(\frac{\xi}{A}\frac{\mathrm{d}A}{\mathrm{d}\xi}\right)_i + \left(\frac{\xi}{s_0}\frac{\partial s_0}{\partial \xi}\right)_i\right]
\end{aligned}\right\} \tag{26}
$$

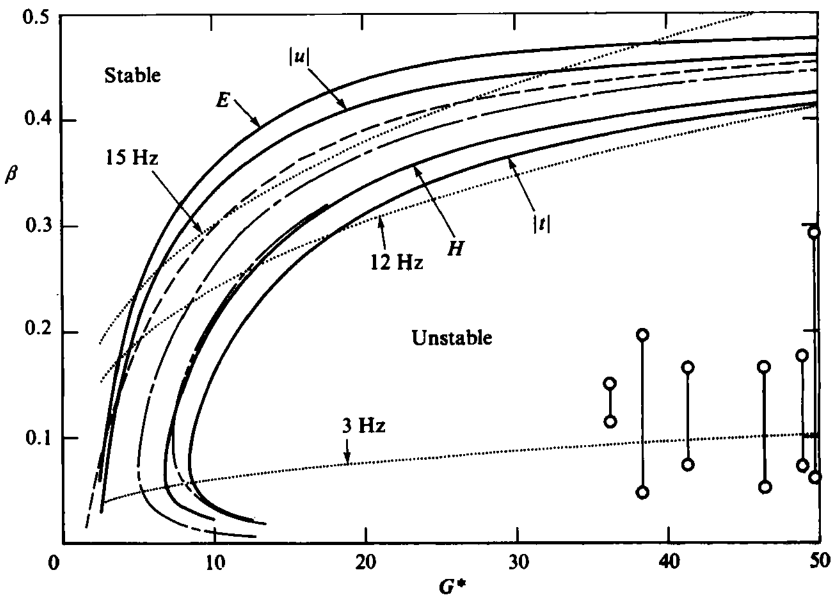

FIGURE 1. Neutral curves based on amplitudes and integral parameters for $Pr = 0.7$: ———, neutral curve according to quasi-parallel theory; ———, neutral curve by Haaland & Sparrow (1973); ————, neutral curve by Hieber & Nash (1975). ...., contours of constant frequencies for air for $Q = 56.3$ W/m (test condition of Pera & Gebhart 1971); ○, data of minimum and maximum frequencies of naturally occurring disturbance temperature in air (Yosinobu *et al.* 1979).

to $O(G^{-1})$, respectively. The leading terms in (26) agree with the wavenumber given by the quasi-parallel theory: α_r. The wavenumber also depends on η and on the disturbance quantity considered.

6. Results

For a range of Grashof numbers and frequencies, the various amplification rates defined in §5 were calculated numerically for the case $Pr = 0.7$ (air). Here the amplification rate for the amplitude in (21) or (22) was evaluated at the point where the amplitude was a maximum. These amplification rates were used to determine the neutral boundaries which separate the stable from the unstable region in the (β, G^*)-plane. Figure 1 shows the neutral curves based on the amplitudes $|u|$ and $|t|$, the kinetic-energy integral E and the thermal-energy integral H, which are measured absolutely, while figure 2 shows the neutral curves based on their relative quantities $|\hat{u}|$, $|\hat{t}|$, $\hat{E}$ and $\hat{H}$. For the sake of comparison, these figures also include the neutral curves, corresponding to $\alpha_i = 0$, according to the quasi-parallel theory, and to the theories of Haaland & Sparrow (1973) and Hieber & Nash (1975).

The curves based on $|u|$ and E in figure 1 lie within the stable region given by the quasi-parallel theory except at extremely low G^*. For $|u|$ and E, therefore, the growth of the boundary layer of the plume leads to a reduction in the stable region. For $|t|$ and H, on the other hand, it leads to an increase in the stable region. Furthermore, the curve based on $|t|$ or H yields critical Grashof numbers of about 6.7 or 8.4, respectively. For such low values of the Grashof numbers two major questions will arise: whether a suitable description of the basic flow is obtained; and whether the approximation to the solutions of the disturbance equations (6) and (7) is valid. The

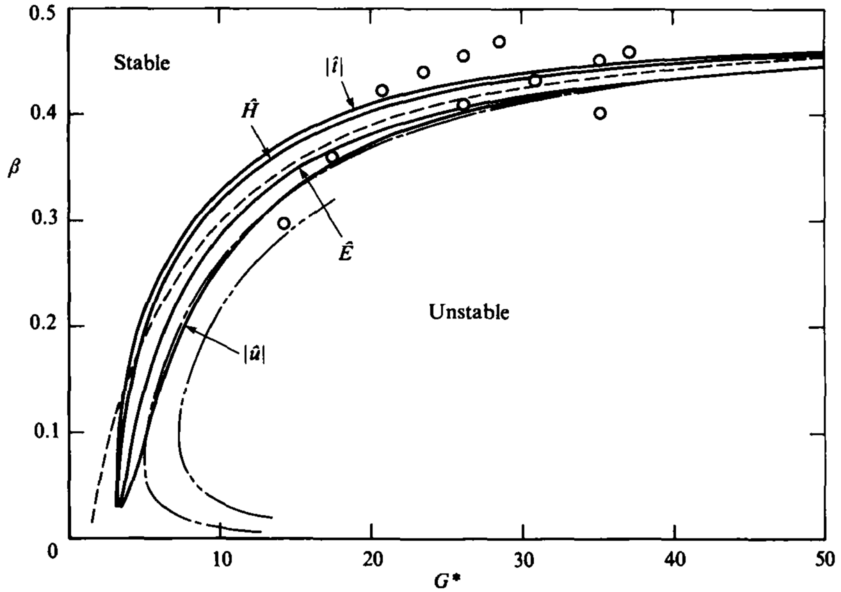

FIGURE 2. Neutral curves based on amplitudes and integral parameters relative to basic flow quantities for $Pr = 0.7$: ----, neutral curve according to quasi-parallel theory; ——, neutral curve by Haaland & Sparrow (1973); ———, neutral curve by Hieber & Nash (1975); ○, data on neutral disturbance temperature in air (Yosinobu *et al.* 1979).

method of approximation used here is valid to $O(G^{*-1})$ for a weakly non-parallel plume. This necessarily requires that $G^* \gg 1$ for a useful result to be obtained. Measurements of the laminar temperature profiles made by Forstrom & Sparrow (1967) show excellent agreement with the analytical predictions based on the usual boundary-layer approximation even at the lowest Grashof number, giving a rough estimate of $G^* \approx 6$. It is also shown by Tatsumi & Kakutani (1958) in their stability analysis of the Bickley jet that the stability characteristics at low Reynolds numbers depend only on integrals of the velocity distribution of the basic flow. The present analysis accounts for higher-order effects of the boundary layer of the plume basic flow. In the light of these facts the stability characteristics obtained at low Grashof numbers will not deviate so far from the exact ones, in comparison with those of Haaland & Sparrow and Hieber & Nash. They do not treat the non-parallel effects of disturbance waves correctly; for example, their analyses do not account for the amplitude function $A(\xi)$ in (9).

The contours of physical constant frequencies, given by $\beta G^{*-\frac{1}{3}} =$ const., are plotted on figure 1 under the test conditions of Pera & Gebhart (1971). They indicated that disturbances with frequencies higher than about 12 Hz were not detected downstream using a hot-wire anemometer. Quantitative comparison of the results obtained here with their observations is not possible since it is not clear whether they actually observed the streamwise component of the disturbance velocity. Yosinobu *et al.* (1979) measured disturbance temperature using a fine thermocouple probe and observed the small disturbances which naturally occur in the region $G^* \approx 40$. Their maximum and minimum frequencies are also plotted on figure 1. These data lie in a region where any disturbance is amplified. Similar results, which are not shown in the figure, are obtained in an experiment by Bill & Gebhart (1975) on transition to turbulence. However, they seem to fail to set a laminar plume stably since the time variation of thermocouple outputs is rather uneven, in comparison with that of Yosinobu *et al.*, even in the laminar region.

Each neutral curve in figure 2, based on the relative quantity, shows only a small deviation from the curve derived from the quasi-parallel theory. This indicates that the neutral curve is strongly affected by the streamwise variations of the basic flow such that the streamwise velocity increases like $x^{\frac{1}{5}}$ and the temperature decreases like $x^{-\frac{3}{5}}$. Thus for the plume flow the use of a relative quantity in the definition of an amplification rate leads to close agreement with the neutral curve obtained from the quasi-parallel theory. Yosinobu *et al.* (1979) made measurements of the growth or decay in the maximum amplitude of the disturbance temperature relative to the basic flow. Their data on neutral disturbances are plotted on figure 2. It should be noted that all of these data would give damped disturbances because of the downstream decrease in the basic temperature, provided the absolute amplitudes were used in measurements of the growth or decay; then, the neutral curve based on $|t|$ in figure 1 will show good agreement. This situation cannot be explained by the theories of Haaland & Sparrow and Hieber & Nash as well as by the quasi-parallel theory.

Figure 3 shows various amplification rates for absolute quantities as a function of β, where $K(u)$ and $K(t)$ are evaluated at η where each amplitude is a maximum. In a higher-frequency region $K(E)$ and $K(u)$ become slightly larger than those obtained from the quasi-parallel theory: $-\alpha_i$. On the other hand, $K(H)$ and $K(t)$ are smaller than $(-\alpha_i)$ over a full frequency range. However, it is noted that every amplification rate obtained in a low-frequency region is smaller than $(-\alpha_i)$. The growth of the plume leads to an increase of stability in the region. This result will be significant in the analysis of downstream developments of disturbances since the region contains the point of a maximum amplification rate.

Figure 4 shows various amplification rates for relative quantities. $K(\hat{E})$ and $K(\hat{t})$ are displayed in figure 4(*b*) and $K(\hat{H})$ in figure 4(*a*); the other curves would be found in the vicinity of those shown. There is no difference between figures 3 and 4 in the amplification rates according to the quasi-parallel theory and to the theories of Haaland & Sparrow and Hieber & Nash. The result indicates that the use of relative quantities leads to small deviations among various amplification rates obtained by the non-parallel theory developed here. Nevertheless, those for the relative quantities are also fairly small compared with $(-\alpha_i)$ in the low-frequency region. Figure 5 shows a comparison of the theoretical amplification rates at $G^* = 20$ with the experimental data of Wakitani & Yosinobu (1984) obtained from measurements of the relative amplitudes of disturbance temperature at $G^* = 19$ using a similar method to that of Dring & Gebhart (1969). Apparently, agreement between experiment and theory has been improved by the first-order correction to the amplification rate. Again, it should be noted that these data were reduced by 0.03 approximately provided that the absolute amplitudes were used.

Figure 6 shows the variations of the amplification rates for u and t with the transverse coordinate η at $G^* = 20$ and 50 while figure 7 shows the variations of the wavenumbers. Both the amplification rate and wavenumber for u exhibit gradual variations with η as shown in these figures. On the other hand, those for t change extensively at large η. The variations of the amplification rate and wavenumber for t are stronger as G^* decreases.

Figure 8 shows the streamwise variation of the wavenumbers based on u and t along two contours of constant frequency. These wavenumbers are also evaluated at η where each amplitude is a maximum. $N(u)$ and $N(t)$ are shown to be nearly equal but to be smaller than those determined from the quasi-parallel theory: α_r. This result

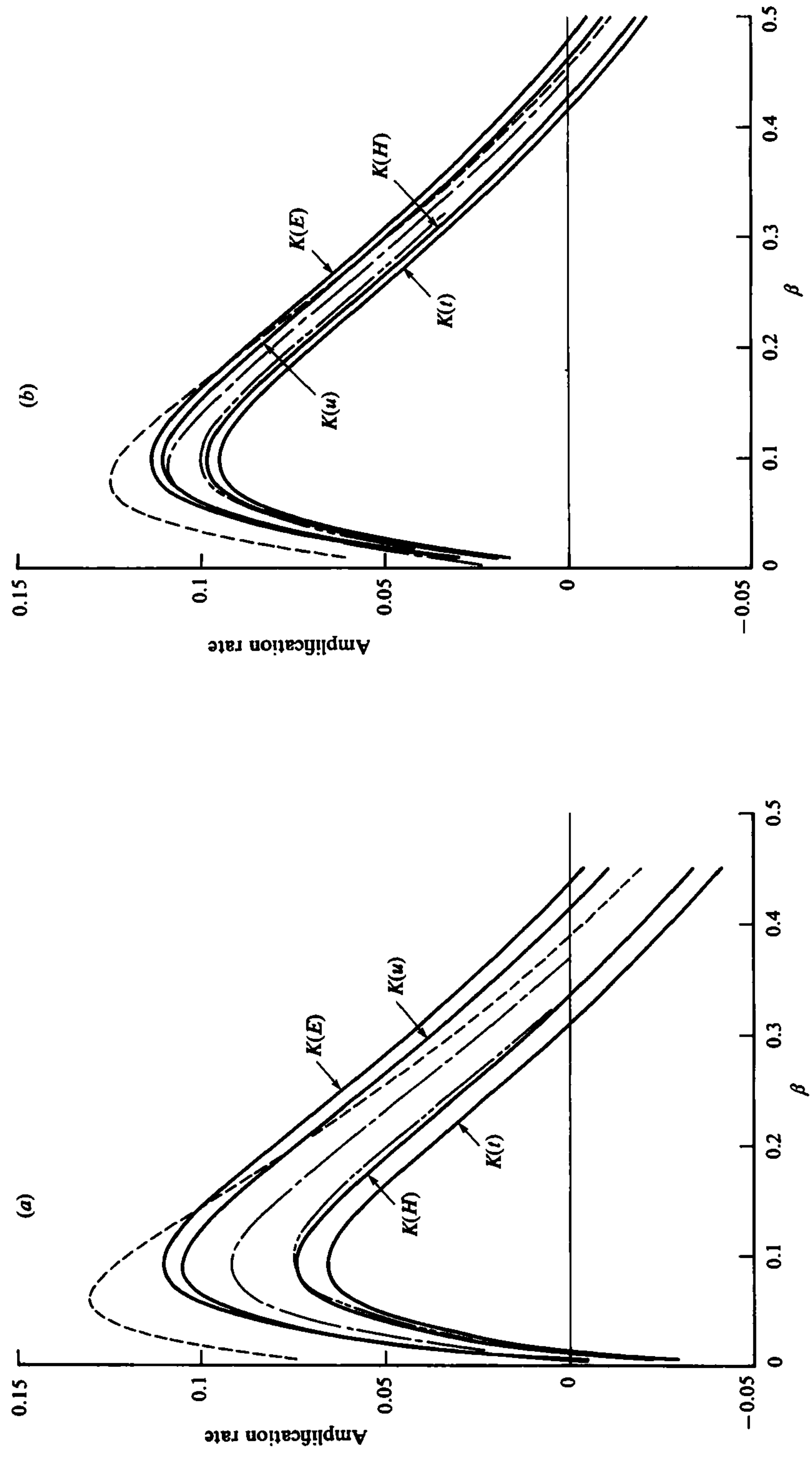

FIGURE 3. Amplification rates for amplitudes and integral parameters as a function of β ($Pr = 0.7$). (a) $G^* = 20$, (b) 50: ——, amplification rate based on quasi-parallel theory ($-\alpha_1$); ——, Haaland & Sparrow (1973); ——, Hieber & Nash (1975).

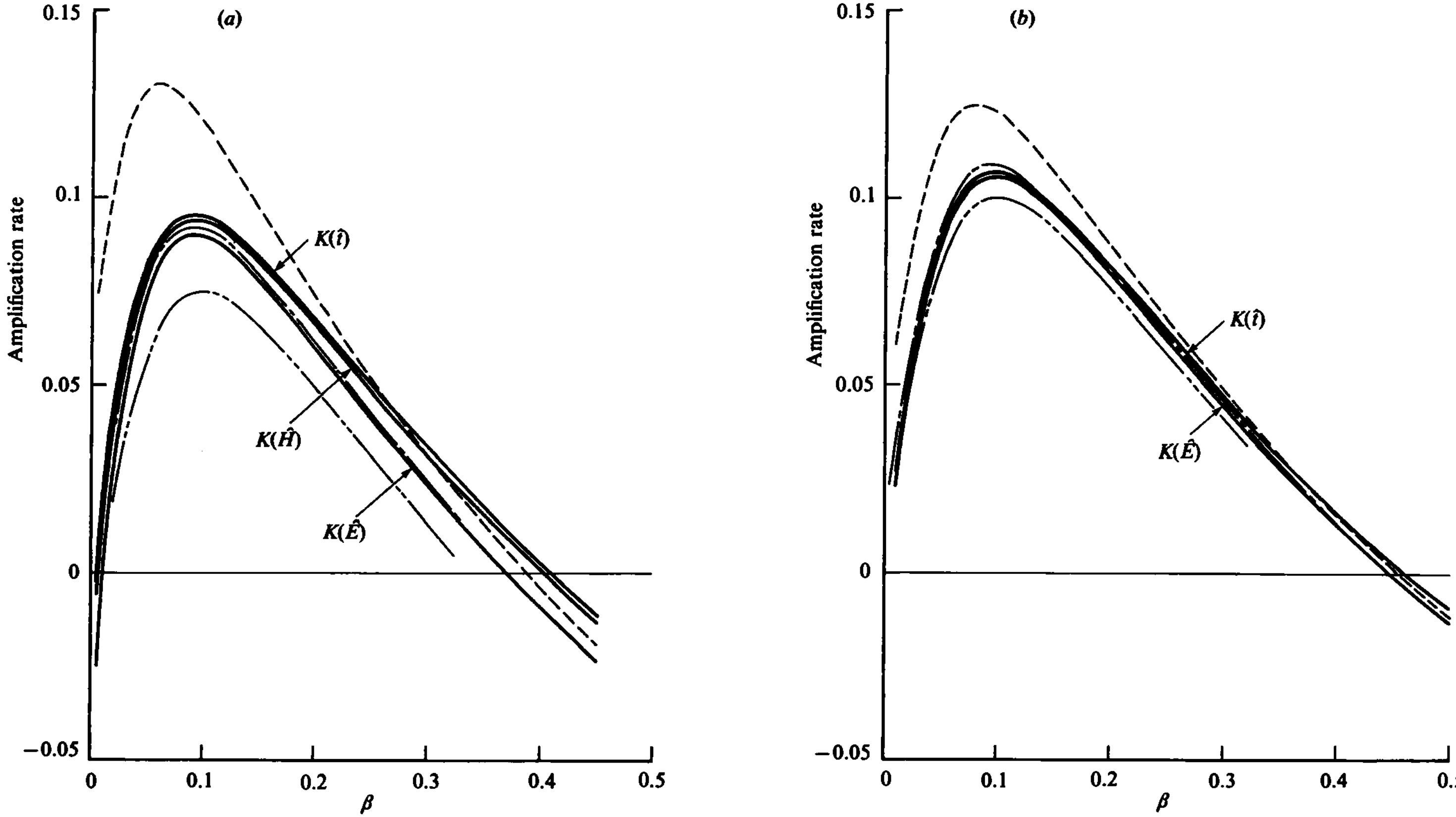

FIGURE 4. Amplification rates for amplitude and integral parameters relative to basic flow quantities as a function of β ($Pr = 0.7$), (*a*) $G^* = 20$, (*b*) 50: ----, $(-\alpha_i)$; ——, Haaland & Sparrow (1973); —·—·—, Hieber & Nash (1975).

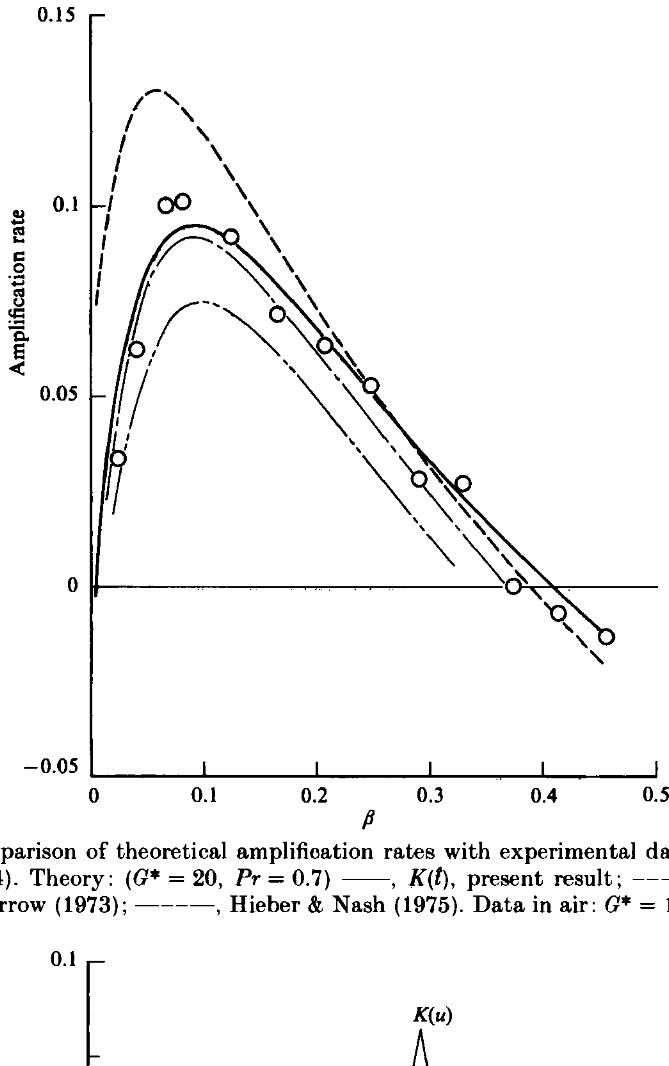

FIGURE 5. Comparison of theoretical amplification rates with experimental data of Wakitani & Yosinobu (1984). Theory: ($G^* = 20$, $Pr = 0.7$) ——, $K(\bar{t})$, present result; ----, $(-\alpha_i)$; ———, Haaland & Sparrow (1973); —·—·—, Hieber & Nash (1975). Data in air: $G^* = 19$, ○.

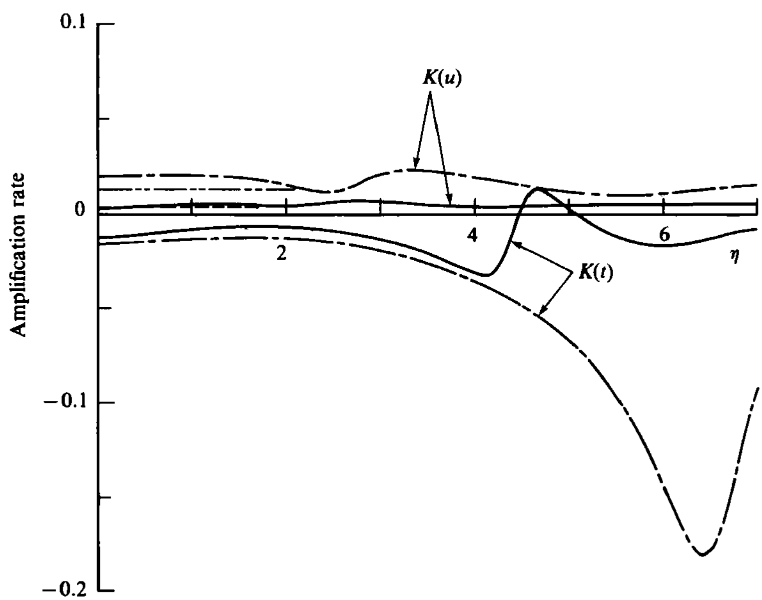

FIGURE 6. Variation of amplification rates for amplitudes with transverse coordinate ($Pr = 0.7$): ——, $G^* = 50$, $\beta = 0.44$; —·—·—, 20, 0.35. $(-\alpha_i)$: ----, $G^* = 50$, $\beta = 0.44$; —··—··—, 20, 0.35.

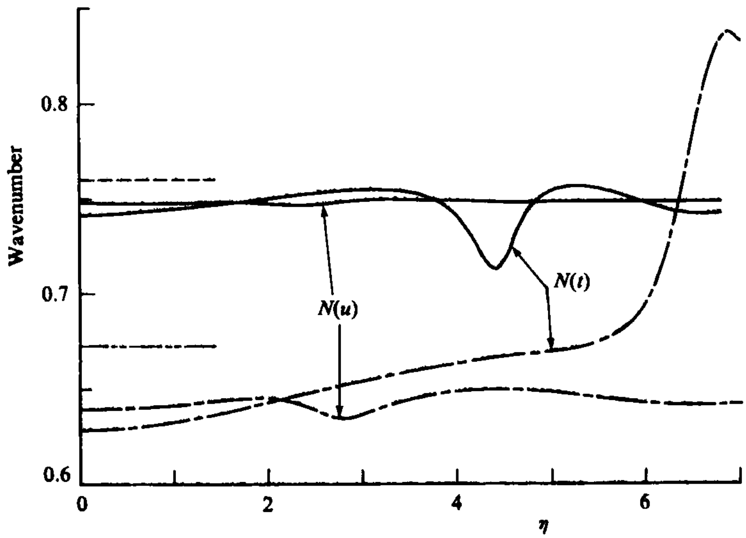

FIGURE 7. Variation of wavenumbers for u-velocity and temperature with transverse coordinate ($Pr = 0.7$). ——, $G^* = 50$, $\beta = 0.44$; — —, 20, 0.35. Wavenumber based on quasi-parallel theory α_r: ----, $G^* = 50$, $\beta = 0.44$; — - —, 20, 0.35.

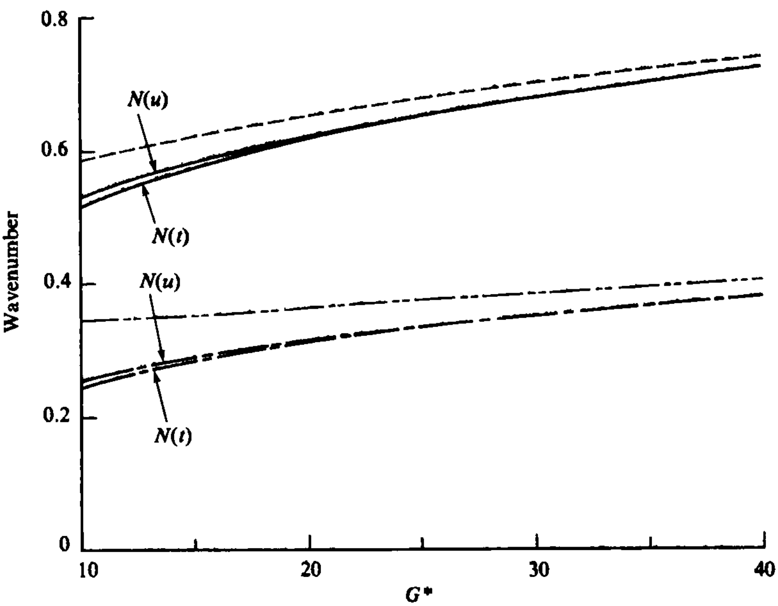

FIGURE 8. Streamwise variation of wavenumbers for u-velocity and temperature along contours of constant frequencies ($Pr = 0.7$). ——, $\beta G^{*-\frac{1}{3}} = 0.1224$; — —, 0.04586. α_r: ----, $\beta G^{*-\frac{1}{3}} = 0.1224$; — - —, 0.04586.

indicates that the first-order correction to the wavenumber as well as the amplification rate is important for the plume flow.

7. Conclusions

The non-parallel-flow effects on the linear stability of a two-dimensional buoyant plume have been investigated for a Prandtl number of 0.7 using the method of multiple scales. The solutions take into account the transverse velocity component, higher-order effects of the boundary layer and the streamwise variations of the basic flow and of the disturbance waves.

The amplification rate (spatial) is a function of the transverse as well as the streamwise coordinate. Furthermore, the amplification rate is a function of the flow quantity involved, i.e. the velocity components, temperature, kinetic energy, etc. This leads to different neutral curves for different flow quantities, in contrast to the quasi-parallel and the other intuitive approaches which have been used previously for the plume. The wavenumber is also a function of the streamwise and the transverse coordinates and the flow quantity.

The neutral curves depend largely on the various flow quantities considered. The shifts in these curves compared with the curve from the quasi-parallel theory are comparatively large over a wide range of Grashof numbers. The use of the flow quantity relative to the basic flow leads to close agreement with the neutral curve obtained from the quasi-parallel theory. This suggests that the streamwise variations of the basic flow largely give rise to the shifts in the neutral curves. The available experimental data support the use of the relative quantity. The quasi-parallel and the other intuitive approaches do not show any difference between the experimental data according to whether the relative quantity has been used.

Each amplification rate within a low-frequency region is smaller than the one predicted by the quasi-parallel theory, even though the relative quantity is used. However, the use of the relative quantities leads to small deviations among various amplification rates according to the non-parallel theory developed here. The non-parallel correction to the wavenumber as well as the amplification rate is significant. The first-order correction to the amplification rate gives good agreement between theory and experiment.

Further experiments to verify the present theory will appear in the near future.

The author is grateful to Professor Hirowo Yosinobu for his valuable discussion and encouragement throughout this work.

Appendix A

Appropriate expansions for the stream function and temperature in the boundary-layer region of the basic flow are given by (4), where f_0 and h_0 are governed by

$$f_0''' + \tfrac{3}{5} f_0 f_0'' - \tfrac{1}{5} f_0'^2 + h_0 = 0, \quad h_0' + \tfrac{3}{5} Pr\, f_0 h_0 = 0, \tag{A 1}$$

$$f_0(0) = f_0''(0) = f_0'(\infty) = 0, \quad \int_0^\infty f_0' h_0 \,\mathrm{d}\eta = \frac{1}{2Pr}. \tag{A 2}$$

The integral condition in (A 2) is a dimensionless representation of the fact that the heat dissipated by the source is convected entirely by the plume. The above is the classical problem described by Fujii (1963) but the integral condition is different.

In the higher-order terms in (4) f_1 and h_1 are governed by

$$\left.\begin{aligned} f_1''' + \tfrac{3}{5} f_0 f_1'' + \tfrac{1}{5} f_0' f_1' + h_1 = 0, \\ h_1'' + Pr\,(\tfrac{3}{5} f_0 h_1' + \tfrac{6}{5} f_0' h_1 + \tfrac{3}{5} f_1' h_0) = 0, \end{aligned}\right\} \tag{A 3}$$

$$f_1(0) = f_1''(0) = h_1'(0) = h_1(\infty) = 0, \quad f_1'(\infty) = \tfrac{3}{5} \cot \frac{2\pi}{5} f_0(\infty). \tag{A 4}$$

The numerical integration of Hieber & Nash (1975) for $Pr = 0.7$ results in

$$\left.\begin{aligned} &f_0'(0) = 0.93273, \quad h_0(0) = 0.49654, \quad f_1'(0) = 0.09969, \\ &h_1(0) = -0.25111, \quad f_0(\infty) = 2.21121, \\ &\quad f_1 \sim f_1'(\infty)\,\eta - 0.38089 \quad \text{as } \eta \to \infty. \end{aligned}\right\} \qquad (\text{A } 5)$$

f_0, f_1, h_0, h_1 and their derivatives can be found easily by solving (A 1)–(A 5) numerically.

Appendix B

To determine $(\xi/A)\,\mathrm{d}A/\mathrm{d}\xi$ from (16) it is necessary to evaluate the Fs from solutions of (10). These quantities are

$$\left.\begin{aligned} F_1 = {} & (2\alpha\beta - 3\alpha^2 f_0' - f_0''')\left(\xi \frac{\partial \phi_0}{\partial \xi} - \tfrac{2}{5}\eta \mathrm{D}\phi_0 + \tfrac{3}{5}\phi_0\right) \\ & + (\beta - 3\alpha f_0')\left(\xi \frac{\mathrm{d}\alpha}{\mathrm{d}\xi} - \tfrac{2}{5}\alpha\right)\phi_0 + \mathrm{i}\alpha[f_1'(\mathrm{D}^2 - \alpha^2) - f_1''']\,\phi_0 \\ & - \tfrac{1}{5}[(f_0'' + 2\eta f_0''')\,\mathrm{D}\phi_0 + (3f_0 - 2\eta f_0')(\mathrm{D}^2 - \alpha^2)\,\mathrm{D}\phi_0] \\ & + f_0'\left[\xi\,\mathrm{D}^2\left(\frac{\partial \phi_0}{\partial \xi}\right) - \tfrac{2}{5}\eta\,\mathrm{D}^3\phi_0 - \tfrac{1}{5}\mathrm{D}^2\phi_0\right], \\ F_2 = {} & (2\alpha\beta - 3\alpha^2 f_0' - f_0''')\,\phi_0 + f_0'\,\mathrm{D}^2\phi_0, \\ F_3 = {} & \xi\left(f_0' \frac{\partial s_0}{\partial \xi} - h_0' \frac{\partial \phi_0}{\partial \xi}\right) + \mathrm{i}\alpha(f_1' s_0 - h_1' \phi_0) \\ & - \tfrac{3}{5}(h_0\,\mathrm{D}\phi_0 + h_0'\phi_0 + f_0\,\mathrm{D}s_0 + f_0' s_0), \\ F_4 = {} & f_0' s_0 - h_0'\phi_0. \end{aligned}\right\} \qquad (\text{B } 1)$$

REFERENCES

Bill, R. G. & Gebhart, B. 1975 The transition of plane plumes. *Intl J. Heat Mass Transfer* **18**, 513–526.

Bouthier, M. 1972 Stabilité linéaire des écoulements presque parallèles. *J. Méc.* **11**, 599–621.

Bouthier, M. 1973 Stabilité linéaire des écoulements presque parallèles. Partie II. La couche limite de Blasius. *J. Méc.* **12**, 75–95.

Dring, R. P. & Gebhart, B. 1969 An experimental investigation of disturbance amplification in external laminar natural convection flow. *J. Fluid Mech.* **36**, 447–464.

Eagles, P. M. & Weissman, M. A. 1975 On the stability of slowly varying flow: the divergent channel. *J. Fluid Mech.* **69**, 241–262.

Forstrom, R. J. & Sparrow, E. M. 1967 Experiments on the buoyant plume above a heated horizontal wire. *Intl J. Heat Mass Transfer* **10**, 321–331.

Fujii, T. 1963 Theory of the steady laminar natural convection above a horizontal line heat source and a point heat source. *Intl J. Heat Mass Transfer* **6**, 597–606.

Garg, V. K. 1981 Spatial stability of the non-parallel Bickley jet. *J. Fluid Mech.* **102**, 127–140.

Garg, V. K. & Round, G. F. 1978 Nonparallel effects on the stability of jet flows. *Trans. ASME* E: *J. Appl. Mech.* **45**, 717–722.

Gaster, M. 1974 On the effects of boundary-layer growth on flow stability. *J. Fluid Mech.* **66**, 465–480.

HAALAND, S. E. 1972 Contributions to linear stability theory of nearly parallel flows. Ph.D. thesis, University of Minnesota.

HAALAND, S. E. & SPARROW, E. M. 1973 Stability of buoyant boundary layers and plumes, taking account of nonparallelism of the basic flows. *Trans. ASME* C: *J. Heat Transfer* **95**, 295–301.

HIEBER, C. A. & GEBHART, B. 1971 Stability of vertical natural convection boundary layers: some numerical solutions. *J. Fluid Mech.* **48**, 625–648.

HIEBER, C. A. & NASH, E. J. 1975 Natural convection above a line heat source: higher-order effects and stability. *Intl J. Heat Mass Transfer* **18**, 1473–1479.

LING, C.-H. & REYNOLDS, W. C. 1973 Non-parallel flow corrections for the stability of shear flows. *J. Fluid Mech.* **59**, 571–591.

MORRIS, P. J. 1981 Stability of a two-dimensional jet. *AIAA J.* **19**, 857–862.

PERA, L. & GEBHART, B. 1971 On the stability of laminar plumes: some numerical solutions and experiments. *Intl J. Heat Mass Transfer* **14**, 975–984.

SARIC, W. S. & NAYFEH, A. H. 1975 Nonparallel stability of boundary-layer flows. *Phys. Fluids* **18**, 945–950.

TATSUMI, T. & KAKUTANI, T. 1958 The stability of a two-dimensional laminar jet. *J. Fluid Mech.* **4**, 261–275.

WAKITANI, S. & YOSINOBU, H. 1984 Stability characteristics of a natural convection flow above a horizontal line heat source. *J. Phys. Soc. Japan* **53**, 1291–1300.

YOSINOBU, H., ONISHI, Y., AMANO, S., ENYO, S. & WAKITANI, S. 1979 Experimental study on instability of a natural convection flow above a horizontal line heat source. *J. Phys. Soc. Japan* **47**, 312–319.